Boundary Value Problem for Equations with a Fractional Derivative and Power-Type Singular Coefficients

Volodymyr Ilkiv[1], Yaroslav Slonovskyi[2], Zinovii Nytrebych[3], Mykhailo Symotiuk[4]

[1,2,3]Lviv Polytechnic National University, Lviv, Ukraine,
[2]Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, Lviv
[1]ilkivvv@i.ua, [2]yaroslav.o.slonovskyi@lpnu.ua, [3]znytrebych@gmail.com

Abstract

The work is devoted to the presentation of the results obtained in the study of a model boundary value problem for the partial differential equation

$$D^{\alpha}_{0+,\,t}u(t,x) = at^{\beta}u_{xx}(t,x), \quad t \in (0,T), \quad a>0, \quad x \in \mathbb{T} \equiv \mathbb{R}/2\pi\mathbb{Z}, \tag{1}$$

$$D^{\alpha-k}_{0+,\,t}\, u(t,x)\big|_{t=0} = b_k(x), \quad k=m+1,\dots,n, \quad x\in\mathbb{T}, \tag{2}$$

$$D^{\alpha-k}_{0+,\,t}\, u(t,x)\big|_{t=T} = b_k(x), \quad k=1,\dots,m, \quad x\in\mathbb{T}, \tag{3}$$

where $\alpha \in (n-1,n)$, $n\in\mathbb{N}$, $\beta > -\alpha$, $1<m<n-1$. The fractional derivative $D^{\alpha}_{0+,\,t}$ with respect to the variable t is defined [1] by means of the Riemann–Liouville fractional integral operator of fractional order by the relations

$$D^{\alpha}_{0+}y(t) = \Big(\frac{d}{dt}\Big)^{n} I^{n-\alpha}_{0+}y(t), \quad t>0, \quad I^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{y(\tau)\,d\tau}{(t-\tau)^{1-\alpha}}, \quad t>0.$$

The functions $b_1(x),\dots,b_n(x)$ and the unknown function $u(t,x)$ are assumed to be 2π-periodic with respect to the variable x. Depending on the values of the parameter α, equation (1) admits different interpretations: as a fractional diffusion equation for $\alpha\in(0,1)$ and as a fractional oscillation equation for $\alpha\in(1,2)$. The time-fractional diffusion-wave equation describes many important physical phenomena in different media [2].

For solving a problem (1)–(3), the representation in the form of a Puiseux–Fourier series was obtained. To estimate the small denominators [3], which arise in the study of the convergence of this series in an appropriate functional space, the Hausdorff fractal dimension [4] was applied.

References

[1] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Amsterdam (1993).

[2] Yu. Povstenko, Linear Fractional Diffusion-Wave Equation for Scientists and Engineers, Birkhäuser (2015).

[3] V. I. Arnold, Small divisor problems in classical and celestial mechanics. *Russian Math. Survey* **18** (1963), 85–191.

[4] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, 3rd Ed., J. Wiley & Sons (2014).