An Elliptic Spectral Problem with Concave and Convex Nonlinearity: Uniqueness and Multiplicity of Positive Solutions

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Abstract

We will discuss the question of ***existence*** and ***multiplicity*** of *positive solutions* to the semilinear elliptic Dirichlet problem

$$-\Delta u = \lambda\, u(x)^{q(x)-1} + f(x,u(x)) \quad \text{for } x \in \Omega\,; \qquad u = 0 \quad \text{on } \partial\Omega\,, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with the boundary of class $C^{1,\alpha}$, $\lambda \in \mathbb{R}$ a spectral parameter, and $f(x,u) = |u|^{r-1}\,u$ is a **signed r-power** ($r > 0$) of the unknown function of (a positive variable) $u \in (0,\infty)$ which depends on the point $x \in \Omega$; $r = q(x) - 1$, for instance.

We will briefly present basic methods for treating the semilinear problem (1) with a ***convex*** and ***concave*** nonlinear reaction $f(x,\,\cdot\,)\colon s \longmapsto |s|^{q(x)-2}s\colon \mathbb{R}_+ \subset \mathbb{R} \to \mathbb{R}$ which (for $s \geq 0$) is ***convex*** in a nonempty open subset $\Omega_+ \stackrel{\text{def}}{=} \{x \in \Omega\colon q(x) > 2\}$ and ***concave*** in another nonempty open subset $\Omega_- \stackrel{\text{def}}{=} \{x \in \Omega\colon q(x) < 2\}$ of a bounded domain $\Omega \subset \mathbb{R}^N$. Here, $\lambda \in \mathbb{R}_+$ is a nonnegative spectral parameter which decides about the existence and multiplicity of positive weak solutions (at least two) to problem (1) in case we take $f \equiv 0$.

The Dirichlet Laplace operator being linear, we observe that the elliptic equation in problem (1) is ***convex*** (***concave***, respectively) at a given point $x \in \Omega$, provided $q(x) \geq 2$ ($1 < q(x) \leq 2$). This is a typical problem with variable powers (exponents).

Finally, if time permits, we will discuss also the classical question of ***uniqueness*** for a related problem with the $p(x)$-Laplacian provided $\sup_{x\in\Omega} q(x) < \inf_{x\in\Omega} p(x)$, holds, and the ***multiplicity*** of large solution branches bifurcating from infinity as $\lambda \searrow 0+$.

References

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