On Time-Dependent Schrödinger Equations Constrained by Measurements

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Abstract

If Schrödinger equation $\mathrm{i}\frac{du}{dt} = Hu$ is constrained to states satisfying $P(t_j)^{\perp}u(t_j) = 0$ for orthogonal projections $P(t_j)$ at discrete times t_j due to ideal measurements, then time evolution is governed by the non-autonomous Hamiltonian equation

$$\mathrm{i}\, du = Hu\, dt - \mathrm{i}P^{\perp}u_{-}\, \mu\,, \tag{1}$$

which could be considered as a generalized time-dependent Schrödinger equation incorporating free evolution as well as evolution under measurements into one equation. Hereby, μ denotes counting measure, the complementary orthogonal projection $P^{\perp}(t_j) = \mathrm{Id} - P(t_j)$ acts at time t_j on the left limit $u_{-}(t_j) := \lim\limits_{t\nearrow t_j} u(t)$, and the non-autonomous part $P^{\perp}u_{-}\,\mu$ may be considered as Lagrangian multiplier determined to satisfy the constraints $P^{\perp}u = 0$ imposed on the system due to measurement. Generalizing the space-time formulation in [1], by applying Nečas theorem [2] we show that for every initial value u_0 in complex Hilbert space there exists a unique càdlàg function u, i.e. u is continuous from the right and has limits from the left in time, solving (1) in an ultra-weak sense.

We further indicate how classical Schrödinger equations modeling non-ideal measurements, where evolution is governed by a particular interaction between measured system and measuring device, tend to generalized Schrödinger equations (1) for the measured system when time of interaction concentrates at an instant. In this way, the measurement problem of quantum mechanics can be solved: Quantum mechanical systems are still modeled by Hamiltonian equations on complex Hilbert spaces to quadratic Hamiltonian functions $\frac{1}{2}\langle Hu, u\rangle$ (inducing vector fields preserving Kähler structure in case of non-constrained free evolution), but in presence of constraints, e.g. due to ideal measurements, dynamics can differ from those governed by classical Schrödinger equation, and non-ideal measurements modeled by such equations converge for the measured system to (1) when time of interaction concentrates at an instant so that the measurement becomes ideal.

References

[1] S. Hain, K. Urban, *An ultra-weak space-time variational formulation for the Schrödinger equation* Journal of Complexity **85** (2024), 101868.

[2] J. Nečas, *Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variation-nelle* Ann. Sc. Norm. Super. Pisa, Cl. Sci. **3**(16) (1962), 305–326.