Beyond the Marginal Stability Criterion: Travelling Waves and Model Discrimination in Reaction–Diffusion Systems

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Abstract

This work investigates the transient dynamics of pattern formation in reaction–diffusion systems in supercritical regime, with particular emphasis on waves emerging from localised perturbations in Turing-type models. While classical studies of diffusion-driven instability have largely focused on stationary patterns, we instead examine pattern formation as an inherently spatio-temporal process. Two complementary analytical approaches are employed: the marginal stability criterion and weakly nonlinear multiple-scales analysis via envelope equations. The former is first critically revisited, highlighting its assumptions and limitations. In contrast, the weakly nonlinear framework provides access not only to the wave speed but also to the profile and amplitude of the propagating front. By comparing the two approaches in the vicinity of the bifurcation point, we show that they yield consistent predictions for the wave speed, while also allowing us to delineate the regime of validity of the marginal stability criterion.

Our findings establish that Turing systems possess a unique characteristic propagation speed that depends on both diffusion and reaction parameters, offering a dynamic counterpart to the traditionally studied pattern wavelength. The analytical predictions are validated against numerical simulations for prototypical reaction kinetics, including Schnakenberg and CDIMA systems. Beyond their theoretical relevance, the results address an increasingly important issue in the application of reaction–diffusion models to real-world systems: model discrimination. Given the flexibility in choosing reaction terms, spatial heterogeneities, domain geometries, and boundary conditions, it is often possible to reproduce observed patterns without uniquely identifying the underlying mechanism, raising concerns about covariance versus causality. In this context, the propagation speed of travelling waves—available here in explicit analytical form—provides a complementary and potentially more discriminating observable alongside the classical pattern wavelength. More broadly, the study offers a systematic framework for analysing front propagation and transient dynamics, contributing to a more rigorous and testable application of reaction–diffusion models in experimental settings.

References

[1] Klika, V., Gaffney, E. A., & Maini, P. K., On the speed of propagation in Turing patterns for reaction–diffusion systems, *Physica D: Nonlinear Phenomena*, 467 (2024), 134268. doi:10.1016/j.physd.2024.134268