Error Bounds for Finite Element Approximations of Stationary Second-Order Mean Field Game Systems

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Abstract

We present error bounds for monotone stabilized finite element discretizations of stationary second-order mean field game (MFG) partial differential inclusion systems of the form

$$\begin{aligned} -\nu\Delta u + H(x,\nabla u) &= F[m](x) && \text{in } \Omega, \\ -\nu\Delta m - G(x) &\in \operatorname{div}(m\partial_p H(x,\nabla u)) && \text{in } \Omega, \end{aligned} \tag{1}$$

supplemented with homogeneous Dirichlet boundary conditions $u = 0$, $m = 0$ on $\partial\Omega$, where the Hamiltonian $H = H(x,p)$ is merely Lipschitz, convex, but (possibly) nondifferentiable with respect to p. Here, $\partial_p H$ denotes the partial subdifferential of H with respect to its second argument, and $\Omega \subset \mathbb{R}^d$ denotes a general bounded polytopal Lipschitz domain. For the system (1) with strongly monotone running cost F and non-negative source G, we prove an asymptotic rate of convergence in the H^1-norm for the approximations of the value function u and in the L^2-norm for the approximations of the player density m, despite the density having minimal regularity *a priori* as a function in H^1 and there being no continuity of the set-valued partial subdifferential $\partial_p H$. The analysis we present jointly leverages recent *a priori* finite element error analysis for MFG systems with differentiable $C^{1,1}$ Hamiltonians [1], and new quantitative results on the approximation of MFG systems with merely $C^{0,1}$ Hamiltonians by MFG systems with regularized $C^{1,1}$ Hamiltonians as the regularization parameter vanishes [2]. This talk is based on joint work with Iain Smears in [3].

References

[1] Y. A. P. Osborne and I. Smears, *Near and full quasi-optimality of finite element approximations of stationary second-order mean field games*, Math. Comp., (2025), https://doi.org/10.1090/mcom/4080.

[2] Y. A. P. Osborne and I. Smears, *Regularization of stationary second-order mean field game partial differential inclusions*, SIAM J. Math. Anal., 57 (2025), pp. 5189–5215, https://doi.org/10.1137/24M1686401.

[3] Y. A. P. Osborne and I. Smears, *Rates of convergence of finite element approximations of second-order mean field games with non-differentiable Hamiltonians*, arXiv preprint arXiv:2506.03039, (2025), https://arxiv.org/abs/2506.03039. (To appear in *Mathematics of Computation*)