**Hölder Regularity for Degenerate First-Order Hamilton-Jacobi Equations Under Controllability Conditions**

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**Abstract**

This is joint work with Alpár Mészáros (Durham University, Durham, England, United Kingdom).

Hamilton-Jacobi(-Bellman) equations are key elements in optimal control and mean field game theory. We study the regularity of solutions to first-order Hamilton-Jacobi equations. Despite the lack of parabolic structure, it is known that viscosity solutions of first-order Hamilton-Jacobi equations are Hölder continuous provided that the Hamiltonian is coercive with respect to the momentum variable with superlinear growth (e.g. [1, 2]). However, a large class of first-order HJB equations do not satisfy this hypothesis, even when the Lagrangian cost in the underlying optimal control problem has superlinear growth with respect to the control variable. This occurs if, in the controlled ODE, certain directions are forbidden to the controller. Concretely, consider a linear time-invariant system

$$\dot{x} = Ax + B\alpha, \qquad x, \alpha \in \mathbb{R}^n \tag{1}$$

where $A$ and $B$ are constant $n \times n$ matrices; if the rank of $B$ is strictly less than $n$, then the Hamiltonian in the HJB equation for an optimal control problem based on this system cannot be coercive in any direction $p$ that is orthogonal to the image of $B$.

Motivated by this type of control problem, we study viscosity solutions $u$ of first-order Hamilton-Jacobi equations of the form

$$\partial_t u + (Ax)^\top D_x u + H(t, x, D_x u) = 0,$$

where $H$ has *degenerate* superlinear growth of the form

$$|B^\top D_x u|^q - f(t,x) \lesssim H(t,x,D_x u) \lesssim |B^\top D_x u|^q + C$$

for some $q > 1$, constant $C \in \mathbb{R}$ and $f$ locally $L^r$ for $r$ sufficiently large. We show that, if the system (1) is *controllable*, then $u$ is Hölder continuous.

# References

[1] P. Cardaliaguet. A note on the regularity of solutions of Hamilton-Jacobi equations with superlinear growth in the gradient variable. *ESAIM Control Optim. Calc. Var.*, 15(2):367–376, 2009.

[2] Chan, C.H., Vasseur, A. De Giorgi Techniques Applied to the Hölder Regularity of Solutions to Hamilton–Jacobi Equations. *In: Gonçalves, P., Soares, A. (eds) From Particle Systems to Partial Differential Equations. PSPDE 2015.* Springer Proceedings in Mathematics & Statistics, vol 209. Springer, Cham, 2017.