Stability and Instability in Anisotropic Multi-Phase Interface Evolution

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Abstract

Interfacial motion driven by diffusion plays a fundamental role in materials science and phase separation phenomena. A prototypical example is the Mullins–Sekerka problem, describing the evolution of interfaces as a gradient flow of surface energy coupled to a quasi-static diffusion field. While the two-phase isotropic Mullins–Sekerka flow has been extensively studied, far fewer results are available for multi-phase configurations with junctions, and even fewer in anisotropic settings. Recent work introduced structure-preserving parametric finite element methods for isotropic two- and multi-phase Mullins–Sekerka problems, enabling robust numerical simulation of evolving interface networks with triple junctions [1, 2]. These schemes are unconditionally stable, exactly volume-conserving, and faithfully reproduce the underlying variational structure of the continuous model, providing a reliable computational foundation for diffusion-driven multi-phase dynamics.

Building on this approach, we develop a systematic computational framework for anisotropic multi-phase interfacial motion, focusing on the anisotropic multi-phase Mullins–Sekerka problem. The model can be viewed as a degenerate multi-phase Stefan-type problem incorporating anisotropic surface energies and mobilities, kinetic undercooling, and force balance conditions at triple junctions. We propose a fully discrete parametric finite element scheme extending the structure-preserving methodology of [1, 2] to the anisotropic setting [3]. The scheme satisfies a discrete energy dissipation property. Numerical experiments in two and three dimensions demonstrate robustness for complex evolving networks, including strongly anisotropic energies, nontrivial triple junction dynamics, and characteristic Mullins–Sekerka instabilities.

References

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