On Positivity Preservation of HDG Methods for the Diffusion Equation on Hypergraphs

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Abstract

Structure preserving discretizations of the diffusion equation need to be locally mass conservative and, in the case of nonnegative data, they should also preserve the positivity of the solution. This is particularly important in applications where the solution represents a physical quantity, such as temperature or concentration, which is nonnegative by definition. The most direct approach to obtain local mass conservation for the diffusion equation comprises dual, mixed methods, which discretize the flux explicitly. A way to bypass the saddle-point structure of the corresponding linear system lies in the concept of hybridization, which has been generalized to cover a large class of so-called hybrid discontinuous Galerkin (HDG) methods in [1]. These methods are locally mass conservative not only for classical domains but even if they discretize the diffusion equation posed on graphs or networks of surfaces, which naturally appear as singular limits of thin structures [2]. The positivity preservation of such HDG methods has not been fully established, which is the focus of this contribution.

It will be shown that only a few low-order HDG methods on hypergraphs are positivity preserving. For nonobtuse simplicial meshes, this will be proved for the lowest-order Raviart–Thomas discretization with the stabilization parameter $\tau = 0$ and for piecewise constant approximations of all variables if $\tau = \mathcal{O}(1)$. For general meshes (also nonsimplicial ones), positivity preservation for piecewise constant approximations can be restored if $\tau = \mathcal{O}(1/h)$ is selected with h denoting the mesh width. This choice of τ leads to positivity preservation also for a few other discretizations. However, for most HDG methods, the positivity can be violated for any choice of τ as can be shown by appropriate counterexamples. The theoretical findings will be ilustrated by numerical experiments.

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References

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