A MLMC-VE Method for Uncertainty Quantification of Elliptic PDEs

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Abstract

In this talk, we consider the following stochastic elliptic PDE: find $u\colon \Omega \times D \to \mathbb{R}$ such that, almost surely in Ω,

$$\begin{cases} -\nabla \cdot (a \nabla u) = f & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1}$$

where $D \subset \mathbb{R}^2$ is an open bounded domain, $f \in L^2(D)$ and the diffusion coefficient is a random field $a\colon \Omega \times D \to \mathbb{R}$ satisfying $0 < \operatorname{ess\,inf}_{x \in D} a(\omega, x) \leq a(\omega, x) \leq \operatorname{ess\,sup}_{x \in D} a(\omega, x) < \infty$ a.e. in D, and with realizations in $L^\infty(D)$. We are interested in deriving reliable approximations for the expected value of the solution $\mathbb{E}[u]$, as well as of some quantity of interest (QoI) of the solution $\mathbb{E}[\mathcal{Q}(u)]$, where $\mathcal{Q}$ is a linear functional of the form $\mathcal{Q}(u) = \int_D u\, q\, dx$ for $q \in L^2(D)$.

The deterministic PDE obtained for a fixed realization of the diffusion coefficient $a(\omega, x)$ is discretized by means of the conforming Virtual Element (VE) method [2]. Novel error estimates for the VE approximation of QoI of the solution are proved.

To discretize the integral in probability, we employ the Multi Level Monte Carlo (MLMC) method [3] that considers multiple levels of spatial resolution: the estimator is first computed on a coarse resolution level, and successive correction terms are then added to improve accuracy. We analyze the h-version of the MLMC-VE method, which relies on a sequence of nested polygonal meshes of the domain D. In practice, such sequence is constructed using a mesh-agglomeration strategy [1]. Computable estimators for both $\mathbb{E}[u]$ and $\mathbb{E}[\mathcal{Q}(u)]$ are designed, and error estimates are derived.

Numerical tests that verify the theoretical error estimates and validate the proposed methodology will be presented.

References

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