

Are Nonlinear First Order Caputo Fractional Differential Equations Solvable?

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Abstract

Consider the first order Caputo fractional differential equation (FDE)

$$(D_{C,a+}^{1-\alpha}u)(x) := (I_{a+}^{\alpha}u')(x) = f(x, u(x)) \quad \text{for almost every } x \in [a, b],$$

where $\alpha \in (0, 1)$, I_{a+}^{α} is the Riemann-Liouville fractional integral, u' is the traditional first-order derivative and $f : [a, b] \times [0, \infty) \rightarrow \mathbb{R}$ is a function. The Caputo FDE can be a single equation or a system of equations.

It was claimed in some previous papers that if f satisfies the locally Lipschitz condition in the second variable, then the Caputo FDE has a unique solution. However, the result would be incorrect, see the open question below.

The above result has been widely used in the literature to obtain the existence and uniqueness of solutions of a variety of models such as disease models and predator-prey models published, for example in **Scientific Reports, PLoS One, Epidemics, Communications in Nonlinear Science and Numerical Simulation**. However, these previous results which applied the above claimed result to obtain the existence and uniqueness of solutions of the models would not be correct unless one can prove that the locally Lipschitz condition implies the necessary condition for the Caputo FDE to have solutions:

$$Fu \in I_{a+}^{\alpha}(L^1[a, b]) \quad \text{for all } u \in S,$$

where S is a ball in $C_+[a, b]$ and $(Fu)(x) = f(x, u(x))$ for almost every $x \in [a, b]$.

The open question is under what conditions on the nonlinearity f , does the above necessary condition hold?

It is noted that if the nonlinearity f satisfies the locally Lipschitz condition in the second variable or is infinitely differentiable, it is unknown whether f satisfies the necessary condition.