Computer Assisted Study of a Perturbed Planar Vector Field in the Quadratic Case

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Abstract

Determining how many limit cycles a planar polynomial system of differential equations can have is a remarkably hard problem. In this work, we study the weak infinitesimal Hilbert 16th problem which asks for the number of limit cycles can a polynomial vector field of degree n have if it is close to a Hamiltonian vector field.

We consider cubic Hamiltonians in the normal form

$$H(x,y)=\frac{1}{2}(x^2+y^2)-\frac{1}{3}x^3+axy^2+\frac{1}{3}by^3.$$

A perturbed system is defined as

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{\partial H(x,y)}{\partial y}+\varepsilon f(x,y),\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\partial H(x,y)}{\partial x}+\varepsilon g(x,y),$$

where f and g are polynomials of degree 2. Let $\gamma_h\subset\{(x,y)\colon H(x,y)=h\}$ with $h\in(0,1/6)$ be an oval surrounding the origin.

In this work, we present a method to carry out a computer assisted proof that for fixed (a,b) the number of limit cycles bifurcating from ovals γ_h for $h\in(0,1/6)$ is less than three. The method is based on the relation between the number of such limit cycles and the number of isolated zeros of the Abelian integrals $I(h)=\oint_{\gamma_h} f(x,y)\mathrm{d}y-g(x,y)\mathrm{d}x$ for $h\in(0,1/6)$.