Oscillation results for first order neutral delay differential equations with several positive and negative coefficients

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Abstract

We provide sufficient criteria for the oscillation of all solutions of neutral delay differential equations of the form

$$\left[x(t) - \sum_{i=1}^{N_r} R_i(t)x(t - r_i(t))\right]' + \sum_{i=1}^{N_p} P_i(t)x(t - \tau_i(t)) - \sum_{i=1}^{N_q} Q_i(t)x(t - \delta_i(t)) = 0,$$

with both positive and negative terms and time-variable delays. Our results improve and generalize several existing criteria available in the literature that address restricted cases, such as constant delays or the absence of negative coefficients. Under additional assumptions on slowly varying parameters, we derive sharper oscillation conditions. We demonstrate the applicability of our findings through illustrative examples.

This is a joint work with Ábel Garab (University of Szeged)

References

[1] R. P. Agarwal, M. Bohner, and W.-T. Li, *Nonoscillation and Oscillation. Theory for Functional Differential Equations*, volume **267** of *Pure Appl. Math., Marcel Dekker*. New York, NY: Marcel Dekker, 2004.

[2] E. R. Attia, On the upper bounds for the distance between zeros of solutions of a first-order linear neutral differential equation with several delays. *AIMS Math.* **9** (9): 23564–23583, 2024.

[3] E. R. Attia and B. M. El-Matary, New results for the upper bounds of the distance between adjacent zeros of first-order differential equations with several variable delays. *J. Inequal. Appl.* **2023**: Article No. 103.

[4] I. Balázs and T. Krisztin, Global stability for price models with delay. *J. Dyn. Differ. Equations* **31** (3):1327–1339, 2019.

[5] C. T. H. Baker, G. A. Bocharov, C. A. H. Paul, and F. A. Rihan, Modelling and analysis of time-lags in some basic patterns of cell proliferation. *J. Math. Biol.* **37** (4): 341–371, 1998.

[6] G. E. Chatzarakis and I. Jadlovská, Improved iterative oscillation tests for first-order deviating differential equations. *Opuscula Math.* **38** (3): 327–356, 2018.

[7] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther, *Delay equations. Functional-, Complex-, and Nonlinear Analysis.*, volume **110** of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1995.

[8] J. G. Dix, Improved oscillation criteria for first-order delay differential equations with variable delay. *Electron. J. Differ. Equ.* **2021**: No. 32, 1–12, 2021.

[9] L. Erbe, *Oscillation Theory for Functional Differential Equations*. Routledge, 1st edition, 1995.

[10] T. Erneux, *Applied Delay Differential Equations*, volume **3** of *Surv. Tutor. Appl. Math. Sci.* New York, NY: Springer, 2009.

[11] E. M. Elabbasy, T. S. Hassan, and S. H. Saker, Oscillation and nonoscillation of nonlinear neutral delay differential equations with several positive and negative coefficients. *Kyungpook Math. J.* **47** (1): 1–20, 2007.

[12] Á. Garab, V. Kovács, and T. Krisztin, Global stability of a price model with multiple delays. *Discrete Contin. Dyn. Syst.* **36** (12): 6855–6871, 2016.

[13] Á. Garab and I. P. Stavroulakis, Oscillation criteria for first order linear delay differential equations with several variable delays. *Appl. Math. Lett.* **106**: Article No. 106366, 2020.

[14] S. R. Grace, S. S. Negi, and S. Abbas, New oscillatory results for nonlinear delay dynamic equations with super-linear neutral term. *Appl. Math. Comput.* **412**: Article No. 126576, 2022.

[15] K. Gopalsamy, X.-Z. He, and L. Wen, On a periodic neutral logistic equation. *Glasg. Math. J.* **33** (3): 281–286, 1991.

[16] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations: with Applications*. Oxford Math. Monogr. Oxford: Clarendon Press, 1991.

[17] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, volume **99** of *Appl. Math. Sci.* New York, NY: Springer-Verlag, 1993.

[18] C. Huang, X. Ding, and Q. Wang, Asymptotic stability analysis on a neutral Nicholson's blowflies equation with time-varying delays. *Electron. J. Qual. Theory Differ. Equ.* **2025**: No. 26, 1–15, 2025.

[19] C. Huang, X. Ding, and Q. Wang, Positive periodic stability to a neutral Nicholson's blowflies model. *Qual. Theory Dyn. Syst.* **24** (3): Article No. 140, 2025.

[20] F. Kádár and G. Stépán, An implicit system of delay differential algebraic equations from hydrodynamics. *Electron. J. Qual. Theory Differ. Equ.* **2023**: Article No. 28, 2023.

[21] R. Koplatadze, Oscillation criteria for first-order differential equations with delay argument. *Semin. I. Vekua Inst. Appl. Math., Rep.* **50**: 35–40, 2024.

[22] Y. Kuang, *Delay Differential Equations: with Applications in Population Dynamics*, volume **191** of *Math. Sci. Eng.* Boston, MA: Academic Press, Inc., 1993.

[23] G. S. Laddem, V. Lakshmikantham, and B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*. Marcel Dekker, Inc., 1987.

[24] B. Li, Oscillation of first order delay differential equations. *Proc. Am. Math. Soc.* **124** (12): 3729–3737, 1996.

[25] Z. Luo and J. Shen, Oscillation and nonoscillation of neutral differential equations with positive and negative coefficients. *Czech. Math. J.* **54** (1): 79–93, 2004.

[26] A. Matsumoto and F. Szidarovszky, Delay differential nonlinear economic models. In *Nonlinear dynamics in economics, finance and the social sciences. Essays in honour of J. B. Rosser Jr. Selected papers of the 5th workshop on dynamic models in economics and finance (MDEF), Urbino, Italy, September 25–27, 2008*, pp. 195–214. Dordrecht: Springer, 2010.

[27] G. M. Moremedi and I. P. Stavroulakis, A survey on the oscillation of delay equations with a monotone or non-monotone argument. In *Differential and difference equations with applications. ICDDEA, Amadora, Portugal, June 5–9, 2017*, pp. 441–461. Cham: Springer, 2018.

[28] N. Parhi and S. Chand, On forced first order neutral differential equations with positive and negative coefficients. *Math. Slovaca* **50** (1): 81–94, 2000.

[29] S. Pinelas and S. S. Santra, Necessary and sufficient condition for oscillation of nonlinear neutral first-order differential equations with several delays. *J. Fixed Point Theory Appl.* **20** (1): Article No. 27, 2018.

[30] M. Pituk, Oscillation of a linear delay differential equation with slowly varying coefficient. *Appl. Math. Lett.* **73**: 29–36, 2017.

[31] M. Pituk and J. I. Stavroulakis, The first positive root of the fundamental solution is an optimal oscillation bound for linear delay differential equations. *J. Math. Anal. Appl.* **507** (1): Article No. 125789, 2022.

[32] M. Pituk, I. P. Stavroulakis, and J. I. Stavroulakis, Explicit values of the oscillation bounds for linear delay differential equations with monotone argument. *Commun. Contemp. Math.* **25** (3): Article No. 2150087, 2023.

[33] V. Răsvan, Absolute stability of a class of control systems described by functional differential equations of neutral type. Equations differ. fonctionnelles non lin., Actes Conf. internat. "Equa-Diff 73", Bruxelles et Louvain-la-Neuve 1973, pp. 381–396, 1973.

[34] V. Răsvan, Critical cases in neutral functional differential equations, arising from hydraulic engineering. *Opuscula Math.* **42** (4): 605–633, 2022.

[35] V. Răsvan, D. Danciu, and D. Popescu, On some neutral functional differential equations occurring in synchronization. In *Delays and interconnections: methodology, algorithms and applications. Selected contributions of the 4th DelSys workshop, Gif-sur-Yvette, France, November 25–27, 2015*, pp. 19–32. Cham: Springer, 2019.

[36] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, volume **57** of *Texts Appl. Math.* New York, NY: Springer, 2011.

[37] I. P. Stavroulakis, A survey on the oscillation of differential equations with several deviating arguments. *J. Inequal. Appl.* 2014: Article No. 399, 2014.

[38] Z. Zhuang, Q. Wang, and H. Wu, A partly sharp oscillation criterion for first-order delay differential equations. *Appl. Math. Comput.*, **409**: Article No. 126422, 2021.

[39] Z. Zhuang, Q. Wang, and H. Wu, A new oscillation criterion for first-order delay differential equations by iteration. *Appl. Math. Comput.* **390**: Article No. 125632, 2021.

[40] L. Zhang and G. Stépán, Hopf bifurcation analysis of scalar implicit neutral delay differential equation. *Electron. J. Qual. Theory Differ. Equ.*, 2018: Article No. 62, 2018.