The Linearized Westervelt–Pennes Bioheat System: Numerical Analysis and Simulation

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Abstract

Ultrasounds have been a staple in clinical diagnostic for decades. More recently, its mechanical and thermal effects have led to its use in several medical applications, through high intensity focused ultrasounds (HIFU), see [1, 2]. We present a robust and accurate numerical methodology for a coupled system of partial differential equations arising in the modelling of HIFU. The system combines the linearized Westervelt equation for acoustic propagation,

$$p_{tt} + \alpha_1 p_t = \alpha_2 \Delta p + \alpha_3 \Delta p_t + f, \tag{1}$$

with the Pennes bioheat equation governing temperature evolution induced by acoustic absorption,

$$T_t + \gamma_1 T = \gamma_2 \Delta T + Q(p^2, p_t^2, \langle p^2 \rangle, \langle p_t^2 \rangle) + g. \tag{2}$$

The heat source in the bioheat equation is allowed to depend on the acoustic field through p^2 and p_t^2, or suitable time-averages of these functions, thus capturing relevant physical regimes.

The proposed scheme admits two equivalent interpretations: a finite difference discretization and a fully discrete, piecewise linear finite element method in space. The scheme is based on nonuniform finite differences in space and employs a uniform discretization in time. We establish second-order convergence in space for the discrete time derivative and gradient of the pressure and the discrete temperature, and first-order convergence in time, even in the presence of nonlinear heat sources driven by the acoustic pressure. The convergence analysis is based on suitable discrete counterparts of the L^p, $p \geq 1$, and H^1 norms and relies on Bramble-Hilbert type arguments, allowing us to avoid standard high-regularity assumptions on the exact solution.

We will present numerical experiments that confirm the theoretical convergence rates as well as a converge rate drop when the use less regular solutions. We also illustrate the qualitative behavior of the acoustic and thermal fields in a HIFU application.

References

[1] Samaneh Haddadi and Mohammad Taghi Ahmadian. Analysis of nonlinear acoustic wave propagation in HIFU treatment using Westervelt equation. *Scientia Iranica*, pages 2087–2097, September 2017.

[2] Gail ter Haar and Constantin Coussios. High intensity focused ultrasound: Physical principles and devices. *International Journal of Hyperthermia*, 23(2):89–104, 2007.