Modular Topology Approach to the Non–Homogeneous Dirichlet Problem for the $p(x)$-Laplacian

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, and let $p : \Omega \to (1, \infty)$ be measurable with

$$p^- := \operatorname*{ess\,inf}_{x \in \Omega} p(x) > n, \qquad p^+ := \operatorname*{ess\,sup}_{x \in \Omega} p(x) = \infty.$$

For boundary data $\varphi \in W^{1,p(\cdot)}(\Omega)$, we consider the non-homogeneous Dirichlet problem for the $p(x)$-Laplacian,

$$\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = 0 \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega.$$

In the regime $p^+ = \infty$, norm-based methods and the classical space $W_0^{1,p(\cdot)}(\Omega)$ are not well suited to this problem. To overcome this difficulty, we work with the modular topology and introduce the space $V_0^{1,p(\cdot)}(\Omega)$, defined as the modular closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Within this framework, we establish existence, and under natural assumptions also uniqueness, of a weak solution u such that

$$u - \varphi \in V_0^{1,p(\cdot)}(\Omega).$$

The solution is obtained as a minimizer of the Dirichlet energy in the modular setting.

The talk explains why the modular approach is more effective than the norm-based one when the Δ_2-condition fails and $p(\cdot)$ is unbounded. In particular, we show that although the norm topology need not be uniformly convex, an appropriately defined modular topology may still possess uniform convexity.

References

[1] M. A. Khamsi, J. Lang, O. Méndez, A. Nekvinda, *The non-homogeneous Dirichlet problem for the $p(x)$-Laplacian with unbounded $p(x)$ on a smooth domain*, J. Differential Equations 434 (2025), 113316.

[2] M. A. Khamsi, J. Lang, O. Méndez, *Modular topologies on vector spaces and their applications* (manuscript/preprint).