On the Characteristic Form of $\mathfrak{g}$-Valued Zero-Curvature Representations

Jiřina Jahnová

Mathematical Institute, Silesian University in Opava, Na Rybníčku 1,
746 01 Opava, Czech Republic
Jirina.Jahnova@math.slu.cz

Abstract

Zero-curvature representations (ZCRs) offer a geometric way of encoding nonlinear partial differential equations by means of auxiliary linear problems. In favorable cases, the equation can be reformulated as the flatness condition of a Lie algebra–valued connection. This viewpoint has proved particularly fruitful in the theory of integrable systems, where it leads to systematic constructions of symmetries, conservation laws, and related structures. In this talk, based on our recent preprint [1], we study $\mathfrak{g}$-valued ZCRs of partial differential equations in two independent variables from the viewpoint of their extension to the infinite jet space, focusing on their characteristic elements.

The notion of a characteristic element for $\mathfrak{g}$-valued ZCRs was introduced in a cohomological context by Marvan [2]. In the abelian case $\mathfrak{g} = \mathbb{R}$, this element coincides with the generating function of a conservation law. In the theory of conservation laws, a crucial step is the possibility of extending the generating function to the full jet space so that the conservation law assumes its characteristic form. This mechanism underlies effective computational approaches. We show that an analogous phenomenon holds for general matrix Lie algebras $\mathfrak{g} \subset \mathfrak{gl}(n)$. More precisely, every $\mathfrak{g}$-valued ZCR extended to the infinite jet space admits a characteristic representative in which the Maurer–Cartan equation takes a characteristic form generalizing the classical one known for conservation laws. This form is preserved under gauge transformations and may therefore be regarded as a natural normal form of a ZCR. As a consequence, we obtain a new necessary condition, independent of the Maurer–Cartan equation, that must be satisfied by any characteristic representative, as well as a sufficient condition for a function to serve as the characteristic element of a given ZCR. In contrast to the abelian case, the additional condition is genuinely restrictive for nonabelian Lie algebras. These results suggest a refinement of computational approaches to ZCRs, in analogy with the interplay of necessary and sufficient conditions in the algorithmic theory of conservation laws.

References

[1] J. Jahnová, *On the characteristic form of $\mathfrak{g}$-valued zero-curvature representations,* arXiv:2508.01224 (2025).

[2] M. Marvan, *On zero-curvature-representations of partial differential equations*, in: Differential Geometry and Its Applications, Proc. Conf. Opava, Czechoslovakia, Aug. 24?28, 1992, Math. Publ. 1 (Silesian University, Opava, 1993) 103–122.