Oseen Resolvent System with Traction Boundary Conditions

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Abstract

Let Ω denote an exterior domain in $\mathbb{R}^3$ with connected C^2-boundary denoted by $\partial\Omega$. Let $p \in (1,\infty)$, $\tau \in \mathbb{R}\backslash\{0\}$ and $\lambda \in \mathbb{C}\backslash(-\infty,0]$. Consider the Oseen resolvent system

$$-\Delta u + \tau\,\partial_1 u + \lambda u + \nabla\pi = f, \;\; \operatorname{div} u = 0 \quad \text{in } \Omega, \tag{1}$$

under the Neumann-type boundary conditions on $\partial\Omega$:

$$\sum_{k=1}^{3} (\partial_j u_k + \partial_k u_j - \delta_{jk}\,\pi - \tau\,\delta_{1k}\,u_j/2)\,n_k^{(\Omega)} = b_j \tag{2}$$

for $1 \le j \le 3$, $f \in L^p(\Omega)^3$ and $b \in W^{1-1/p,\,p}(\partial\Omega)^3$. Here $n^{(\Omega)}$ denotes the outward unit normal to Ω. We look for solutions $(u,\,\pi)$ to (1), (2) with $u \in W^{2,p}(\Omega)^3$, $\pi \in D^{1,p}(\Omega)$ and $u(x) \to 0$ for $|x| \to \infty$, where $D^{1,p}(\Omega)$ denotes the set of all functions $\sigma \in W^{1,1}_{loc}(\Omega)$ such that $\nabla\sigma \in L^p(\Omega)^3$. (Functions from this set belong to $W^{1,p}(B_R \cap \Omega)$ for any $R > 0$ with $\mathbb{R}^3\backslash\Omega \subset B_R$.) We further require that at least one of the two side conditions $\int_{\partial\Omega} u \cdot n^{(\Omega)}\,do_x = 0$ or $\pi \in L^s(\Omega)$ for some $s \in (1,\infty)$ are fulfilled. Moreover we are interested in the estimate

$$|\lambda|\,\|u\|_p + \|u\|_{2,p} + \|\nabla\pi\|_{1,p} \le \mathfrak{C}\,(\|f\|_p + |\lambda|^{1/2-1/(2p)}\,\|b\|_{1-1/p,\,p}) \tag{3}$$

for f and b as above, provided that either $(\Im\lambda)^2/\tau^2 + \Re\lambda \ge \Lambda_0,\; |\lambda| < \Lambda_1$ or $|\lambda| \ge \Lambda_1,\; |\arg(\lambda)| \le \vartheta$, where $\vartheta \in [0,\pi),\; \Lambda_0,\, \Lambda_1 \in (0,\infty)$ with $\Lambda_0 < \Lambda_1$, The constant $\mathfrak{C}$ in (3) should depend only on Ω, τ, ϑ, Λ_0 and Λ_1.

It turns out that solutions to this boundary value problem exist and in the case $\int_{\partial\Omega} u \cdot n^{(\Omega)}\,do_x = 0$ are unique and verify (3). If it is instead required that $\nabla\pi \in L^s(\Omega)^3$ for some $s > 1$, such a result holds only under certain restrictions. In particular the condition $p > 3/2$ must be imposed.