Functional Equations Involving Reflection and Piecewise Constant Arguments

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Abstract

Differential equations remain a fundamental tool for describing phenomena across mathematics, physics, engineering, and the life sciences. In recent decades, increasing attention has been given to models incorporating nonstandard functional dependence, leading to the development of differential equations with involution and equations with piecewise constant arguments. Equations with involution, characterized by a self-inverse transformation of the independent variable, naturally arise in problems involving symmetry and nonlocal effects [2]. Meanwhile, equations with piecewise constant arguments provide a suitable framework for systems where state variables are evaluated at discrete moments, as occurs in control processes, epidemiology, and economic dynamics [3]. These classes of equations offer a flexible setting for modeling hybrid and nonstandard dynamical behaviors.

In this work, we establish the theoretical framework for the study of a general two point n-th order differential equation incorporating both a reflection term $m\,v(-t)$ and a piecewise continuous dependence $M\,v([t])$ [1]. The analysis is carried out using Green's function theory, which allow us to obtain its explicit expression and study its qualitative properties. In particular, we obtain comparison results that allow us to determine regions where the related Green's function maintains a constant sign. We then focus on a periodic first-order equation with reflection and piecewise constant arguments. Once the corresponding Green's function and its sign structure are characterized, we address the existence of solutions for the related nonlinear problem by applying fixed point techniques, with special emphasis on the monotone method.

References

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