The Dirichlet Problem for the Laplacian in Lipschitz Domains

Amrouche Chérif

Université de Pau et des Pays de l'Adour, Pau, France
CHERIF.AMROUCHE@UNIV-PAU.FR

Abstract

We are interested here in the homogeneous and non-homogeneous Dirichlet problem for the Laplacian in bounded Lipschitz domains. Although it has been extensively studied by many authors, we would like to return to a number of fundamental questions and known results, such as the traces and the **maximal regularity** of solutions. First, to treat non-homogeneous boundary conditions, we rigorously define the notion of traces for non regular functions. This approach replaces the non-tangential trace notion that has dominated the literature since the 1980s. We identify a functional space

$$E(\nabla;\,\Omega) \;=\; \Big\{\, v \in H^{1/2}(\Omega);\ \nabla v \in [\boldsymbol{H}^{1/2}(\Omega)]' \,\Big\}, \tag{1}$$

which satisfies the embeddings $H^{1/2}_{00}(\Omega) \hookrightarrow E(\nabla;\,\Omega) \hookrightarrow H^{1/2}(\Omega)$. The trace operator is well-defined and continuous from $E(\nabla;\,\Omega)$ into $L^2(\Gamma)$, leading to a new characterization of $H^{1/2}_{00}(\Omega)$ as the kernel of this operator. Second, we address the regularity of solutions to the Laplace equation with homogeneous Dirichlet conditions. Using specific equivalent norms in fractional Sobolev spaces and Grisvard's results for polygons and polyhedral domains, we prove that maximal regularity $H^{3/2}$ holds in any bounded Lipschitz domain Ω, for all right-hand sides in the dual of $H^{1/2}_{00}(\Omega)$. This conclusion contradicts the prevailing claims in the literature since the 1990s. Third, we describe some criteria which establish new uniqueness results for harmonic functions in Lipschitz domains. In particular, we show that if $u \in H^{1/2}(\Omega)$ or $u \in W^{1,2N/(N+1)}(\Omega)$, with $N \geq 2$, is harmonic in Ω and vanishes on Γ, then $u \equiv 0$. These criteria play a central role in deriving regularity properties. Finally, we revisit the classical Area Integral Estimate of Dahlberg, and of Kenig, Pipher, and Verchota. For a harmonic function u in Ω vanishing at some interior point, the estimate asserts

$$\int_\Gamma |u|^2 d\sigma \leq C \int_\Gamma |S(u)|^2 d\sigma \simeq C \int_\Omega \varrho |\nabla u|^2 dx, \tag{2}$$

where $S(u)$ is the area integral of u and ϱ is the distance to the boundary. Using Grisvard's work in polygons and an explicit function given by Nečas, we show that this inequality cannot hold in its stated form.

References

[1] C. Amrouche and M. Moussaoui. Maximal regularity of Dirichlet problem for the Laplacian in Lipschitz domains, *ArXiv* 2025, https://arxiv.org/abs/2509.08543

[2] C. Amrouche and M. Moussaoui. On the traces of harmonic functions $H^{1/2}$ and $H^{3/2}$ in Lipschitz domains, https://arxiv.org/abs/2601.05610

[3] B.E.J. Dahlberg, C.E. Kenig, J. Pipher and G.C. Verchota. Area integral estimates for higher order elliptic equations and systems. *Ann. Inst. Fourier*, **47-5**, 1425–1461, (1997).

[4] P. Grisvard.: Elliptic Problems in Nonsmooth Domains, Pitman, Boston, (1985).

[5] D. Jerison and C.E. Kenig. The Inhomogeneous Dirichlet Problem in Lipschitz Domains, *J. Funct. Anal.* **130**, 161–219, (1995).

[6] J.L. Lions and E. Magenes. *Problèmes aux limites non-homogènes et applications*, Vol. 1, Dunod, Paris, (1969).

[7] J. Nečas. *Direct methods in the theory of elliptic equations*. Springer Monographs in Mathematics. Springer, Heidelberg, (2012).